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# Outer conjugacy problem of orbit preserving transformations

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We report here on the outer conjugacy problem of orbit preserving transformations, which is very closely related to von Neuman algebra theory. Details will be published in [3].

In 1936 F. Murray and J. von Neuman raised a problem of classifying factors which are von Neuman algebras with the trivial center. As they showed, an example of factors can be constructed from an ergodic automorphism on a Lebesgue space, which is so called a cross product von Neuman algebra. A measurable and invertible mapping  $\phi$  from a  $\sigma$ -finite Lebesgue space  $(\Omega, \mathcal{B}, m)$  onto a  $\sigma$ -finite Lebesgue space  $(\Omega', \mathcal{B}', m')$  is called an isomorphism if  $m'(\phi(E)) = 0$  if and only if  $m(E) = 0$ . An isomorphism of  $\Omega$  onto itself is called an automorphism. Let  $T$  be an automorphism of  $(\Omega, \mathcal{B}, m)$ . The cross product von Neuman algebra  $L^\infty(\Omega) \otimes_{\mathbb{T}}$  is the weak closure of the linear hull of the sets of operators  $U$  and  $L_f$  for  $f \in L^\infty(\Omega)$  acting on the Hilbert space  $L^2(\Omega) \otimes \ell^2(Z)$ , defined by the following: for  $\xi(\omega, n) \in L^2(\Omega) \otimes \ell^2(Z)$

$$U\xi(\omega, n) = \xi(T^{-1}\omega, n-1)((dmT^{-1}/dm)(\omega))^{1/2}$$

$$L_f(\omega, n) = f(\omega)\xi(\omega, n).$$

In ergodic theory isomorphism problems for automorphisms have been studied. On the other hand operator algebrists consider an isomorphism problem for  $*$ -automorphisms of a von Neuman algebra. Let  $M$  be a von Neuman algebra and  $\alpha$  and  $\alpha'$

be  $*$ -automorphisms of  $M$ . It is natural to ask when  $*$ -automorphisms  $\alpha$  and  $\alpha'$  are conjugate, i.e.  $\beta\alpha\beta^{-1} = \alpha'$  for some  $*$ -automorphism  $\beta$  of  $M$ , or when they are outer conjugate, i.e.  $\beta\alpha\beta^{-1} = \alpha'\gamma$  for an inner  $*$ -automorphism  $\gamma$  of  $M$  and a  $*$ -automorphism  $\beta$  of  $M$ . This is called an isomorphism problem in non commutative ergodic theory.

We discuss about this problem on the cross product von Neumann algebra  $L^\infty(\Omega) \otimes \mathbb{Z}$ , which is not abelian. For this we consider orbit preserving transformations (o.p.t.)  $R$  of  $T$ . They are automorphisms of  $\Omega$  satisfying

$$\{RT^i\omega : i \in \mathbb{Z}\} = \{T^i R\omega : i \in \mathbb{Z}\} \text{ a.e. } \omega.$$

If  $R\omega$  is in  $\{T^i\omega : i \in \mathbb{Z}\}$  a.e.  $\omega$  then it is said to be inner. We write

$$N[T] = \{\text{o.p.t.'s of } T\}$$

$$[T] = \{\text{inner o.p.t.'s of } T\}$$

and call them the normalizer group and the full group of  $T$ .

Every o.p.t.  $R$  induces a  $*$ -automorphism  $R$  of the cross product von Neuman algebra  $L^\infty(\Omega) \otimes \mathbb{Z}$  as follows: Let  $RTR^{-1}\omega = T^n\omega$ ,  $\omega \in A_n$ , where  $\{A_n\}_{-\infty < n < \infty}$  is a partition of  $\Omega$ , then  $*$ -automorphism  $R$  is defined by

$$R : U \longmapsto \sum_{-\infty < n < \infty} U^n L_{\chi_{A_n}},$$

and for  $f \in L^\infty(\Omega)$

$$R : L_{f(\omega)} \longmapsto L_{f(R\omega)}.$$

If  $R$  is an inner automorphism of  $T$  then the  $*$ -automorphism  $R$  is inner. Because, since  $R\omega = T^n\omega$ ,  $\omega \in B_n$  for a partition

$\{B_n\}_{-\infty < n < \infty}$  of  $\Omega$ , we have  $R = V \cdot V^{-1}$ , where  $V$  is the unitary element in  $L^\infty(\Omega) \otimes \mathbb{Z}$  defined by

$$V = \sum_{-\infty < n < \infty} U^n L \chi_{B_n}.$$

What we are going to discuss is the following

### Outer conjugacy problem of O.P.T.'s.

We assume that an automorphism  $T$  of  $(\Omega, \mathcal{B}, m)$  is ergodic.  $R$  and  $R'$  in  $N[T]$  are said to be outer conjugate if there is a  $\phi$  in  $N[T]$  such that

$$\phi R \phi^{-1} \in R'[T],$$

or equivalently if the cosets  $R[T]$  and  $R'[T]$  are conjugate in  $N[T]/[T]$ . We remark that in this case  $R$  and  $R'$  are outer conjugate as a  $*$ -automorphism of  $L^\infty(\Omega) \otimes \mathbb{Z}$ .

As an invariant for outer conjugacy one can consider the outer period  $p_0(R)$  of  $R$  in  $N[T]$ . It is the least positive integer  $p$  such that  $R^p$  is in  $[T]$  if it exists. If otherwise, we define  $p_0(R) = 0$  and say that such  $R$  is outer aperiodic. Then it is obvious that the outer period  $p_0(R)$  is an invariant for the outer conjugacy.

When  $T$  has a  $\sigma$ -finite invariant measure  $\mu$  equivalent to  $m$  (in this case we say  $T$  is of type II), for  $R$  in  $N[T]$  the Radon-Nikodym density  $(d_\mu R / d_\mu)(\omega)$  is constant a.e.  $\omega$ , which we denote by  $\text{mod} R$ . Of course if the measure  $\mu$  is finite (in this case we say  $T$  is of type II<sub>1</sub>), then  $\text{mod} R$  is always 1. If  $T$  is of type II, the couple of  $p_0(R)$  and  $\text{mod} R$  is a

complete invariant for the outer conjugacy, which was proved by A. Connes and W. Krieger[1].

When  $T$  has no  $\sigma$ -finite invariant measures equivalent to  $m$  (in this case we say  $T$  is of type III), a complete invariant for outer conjugacy is still unknown. So we think about the conjugate classes of the quotient group  $N[T]/[T]^-$ , which has a close connection with the group  $N[T]/[T]$ , where  $[T]^-$  is the closure of  $[T]$  with respect to the topology defined by the following: For  $R$  in  $N[T]$  the open base of  $R$  is the family of the sets  $\{\phi \in N[T] : \|R \cdot f_i - \phi \cdot f_i\|_{L^1(m)} < \varepsilon, i = 1, 2, \dots, n, \text{ and } m(\omega : RT^i R^{-1} \omega \neq \phi T^i \phi^{-1} \omega) < \varepsilon, i = 0, \pm 1, \dots, \pm n\}$ , where  $f_i \in L^1(\Omega), \varepsilon > 0, R \cdot f(\omega) = f(R^{-1}\omega)(dmR^{-1}/dm)(\omega), f \in L^1(\Omega)$ . We note that  $N[T]$  is a polish group with respect to this topology.

Theorem 1. Let  $T$  be an ergodic automorphism of  $(\Omega, \mathcal{B}, m)$ .

- (1) If  $T$  has a finite invariant measure equivalent to  $m$ , then  $N[T] = [T]^-$ .
- (2) If  $T$  has a  $\sigma$ -finite infinite invariant measure equivalent to  $m$ , or if  $T$  does not admit a  $\sigma$ -finite invariant measure then  $N[T]/[T]^-$  is topologically isomorphic to the centralizer  $c((F_t))$  of the flow  $(F_t)_{t \in \mathbb{R}}$  which determines the weak equivalence class of  $T$ .

Here  $C((F_t))$  is the set of all automorphisms commuting with the flow  $(F_t)$  and the topology of the centralizer is the relative topology of the weak topology on the set of all automorphisms: Let  $(X, \mathcal{F}, \mu)$  be the Lebesgue space on which  $(F_t)$  acts. For an automorphism  $U$  of  $X$ , the open base of  $U$  is

the family of the sets  $\{S: S \text{ an automorphism of } X \text{ such that } \|U \circ f_i - S \circ f_i\|_{L^1(X)} < \varepsilon \text{ } i=1,2,\dots,n\}$   $n=1,2,\dots,\varepsilon > 0, f_i \in L^1(X)$ .

Let us explain about the flow  $(F_t)_{t \in \mathbb{R}}$ . W. Krieger[4] and T. Hamachi-Y. Oka-M. Oshikawa[2] introduced the flow  $(F_t)$  associated with a given ergodic automorphism  $T$  satisfying that if  $T$  on  $(\Omega, \mathcal{B}, m)$  and  $T'$  on  $(\Omega', \mathcal{B}', m')$  are weakly equivalent, i.e. there exists an isomorphism  $\psi$  from  $\Omega$  onto  $\Omega'$  such that  $\psi[T]\psi^{-1} = [T']$ , then the flows  $(F_t)$  and  $(F'_t)$  are isomorphic. Moreover, Krieger proved that this mapping is a one to one and onto mapping from the weak equivalence class of an ergodic automorphism without  $\sigma$ -finite invariant measure to the isomorphism class of an ergodic conservative flow of automorphisms of a Lebesgue space.

It is known that an ergodic automorphism  $T$  has a  $\sigma$ -finite invariant measure if and only if the flow  $(F_t)$  is the translation,  $u \mapsto u+t$  on  $\mathbb{R}$ . We note that in this case the isomorphism between the groups  $N[T]/[T]^-$  and  $C((F_t))$  is given by

$$R \in N[T] \longrightarrow u \longmapsto u + \log(\text{mod } R) \in C((F_t)),$$

where the kernel is  $[T]^- = \{R \in N[T]: \text{mod } R = 1\}$ .

Thus by this theorem there is a one to one and onto map from the conjugate classes of  $N[T]/[T]^-$  to the conjugate classes of  $c((F_t))$ . This is a partial answer to our problem at the moment.

Next, which group appears as the quotient group  $N[T]/[T]$  ?  
For instance we have

Theorem 2. Let  $T$  be an ergodic automorphism without  $\sigma$ -finite invariant measure and  $(F_t)_{t \in \mathbb{R}}$  be the associated flow. Then  $N[T]/[T]$  is compact if and only if  $(F_t)_{t \in \mathbb{R}}$  is measure preserving and has pure point spectrum. In this case  $N[T]/[T]$  is isomorphic to the character group of the  $T$ -set, which is the set of real numbers  $t$  such that the cocycle  $\exp(it \log(dmT/dm)(\omega))$  is a coboundary for  $T$ , i.e. there is a measurable function  $\exp(i\xi_t(\omega))$  such that

$$\exp(it \log(dmT/dm)(\omega)) = \exp(i\xi_t(T\omega)) / \exp(i\xi_t(\omega))$$

a.e. $\omega$ .

Finally it seems to me that the following question is affirmative: Is the couple of outer period and the conjugate class of the centralizer of  $(F_t)_{t \in \mathbb{R}}$  a complete invariant for the outer conjugacy of  $T$  ?

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